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ROBUSTNESS OF THE LMS

When someone talks about robustness, the question that comes to mind is in what sense?

We will show here that LMS is robust in the H^{∞} sense. The H^{∞} norm of a dynamic system is the maximum amplification the system can make to the energy of the input signal. (H stands for Hardy space).

We will show that the LMS is the EXACT solution of a LOCAL optimization problem.

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LEAST PERTURBATION PROPERTY OF LMS

Let w_{i-1} be the weigth estiamte at time i-1 and let $\{d_i, x_i\}$ be the desired and input at time i. Let w_i be a new weight estimate from the available information $\{w_{i-1}, d_i, x_i\}$.

Define the a priori error $\quad e(i) = d(i) - \underline{\underline{W_{i-1}}}x(i)$

and the a posteriori error $\quad r(i) = d(i) - W(i)x(i)$

We seek the w_i that solves the problem

$$\min_{w_i} \|W(i) - W(i-1)\|^2 \quad \text{SUBJECT TO} \quad r(i) = \left(1 - \eta\|x(i)\|^2\right)e(i)$$

The constraint is most relevant for small stepsizes

$$0 < \eta\|X(i)\|^2 < 2 \qquad \forall i$$

In fact under this condition, the a posteriori error is ALWAYS smaller than the a priori error $\quad |r(i)| < |e(i)|$

or $x_i w_i$ is a better fit to d_i than $x_i w_{i-1}$.

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The solution can be obtained as follows . LET $\Delta W = W(i) - W(i-1)$

$$\Delta W . X(i) = W(i)X(i) - W(i-1)X(i)$$
$$= \left[W(i)X(i) - d(i)\right] + \left[d(i) - W(i-1)X(i)\right]$$
$$= -r(i) + e(i)$$
$$= \eta \|X(i)\|^2 e(i)$$

The optimization problem is equivalent to determine an incremental weight of smallest Euclidean norm that satisfies

$$\min_{\Delta W} \|\Delta W\|^2 \quad \text{SUBJECT TO} \quad \Delta W X(i) = \eta \|X(i)\|^2 e(i)$$

The constraint $\Delta W X(i) = \eta \|X(i)\|^2 e(i)$

amounts to an under determined linear system of equations in Δw (infinitly many solutions), but we are interested in the one with the smallest Euclidean norm. The non trivial solution is when $\|X(i)\|^2 \neq 0$

In this case

$$\Delta W^* = \eta X(i) e(i)$$

(MULTIPLY BY X(i) FROM THE LEFT)

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Now substituting we get

$$w(i) = w(i-1) + \eta\, x(i)\left[d(i) - x(i)\, w(i-1)\right]$$

This is the LMS update equation, i.e. the LMS is an optimal solution for the minimization of local perturbations.

Note that there are infinitely many solutions to $\Delta w\, x(i) = \eta \|x(i)\|^2 e(i)$ because if we add a positive constant to the LMS weight we still get a solution

$$x(i)\left[\Delta w^* + z\right] = \eta \|x(i)\|^2 e(i)$$

Z MUST SATISFY $x(i)\, z = 0$

i.e. any vector in the null space of x_i will also be a solution. But it is easy to show that all have a larger norm

$$\|\Delta w^* + z\|^2 = \|\Delta w^*\|^2 + \|z\|^2 + \underbrace{\cancel{\Delta w^* z}}_{0} + \underbrace{\cancel{z\, \Delta w^*}}_{0}$$

$$> \|\Delta w^*\|^2$$